Approximate quantum data storage and teleportation

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Abstract

In this paper we present an optimal protocol by which an unknown state on a Hilbert space of dimension N can be approximately stored in an M-dimensional quantum system or be approximately teleported via an M-dimensional quantum channel. The fidelity of our procedure is determined for pure states as well as for mixed states and it is compared with theoretical results for the maximally achievable fidelity. Results are also given for the fidelity of teleportation of states which are entangled with auxiliary quantum systems of varying Hilbert space dimension.

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1 Introduction

Imagine the following scenario: We are given an unknown quantum state of an N-level system, and we want to store that state, but as storage medium we have only a classical storage device and a physical M-level quantum system ($M < N$). What is the optimal protocol for this task, and with which probability will we be able to retrieve the original state after the process? This problem is formally equivalent to the one of transporting an unknown state, but having as transport medium only a classical channel and an M-level quantum channel, either in the form of a portable quantum system with M levels or a teleportation channel with an initially prepared entangled state of the form $\frac{1}{\sqrt{M}}\sum_{i=1}^{M}|i_A\rangle|i_B\rangle$.

In the present paper we shall present a protocol to achieve these goals with a mean fidelity (to be defined below) for a pure input state of $F = (M+1)/(N+1)$. And we shall show that the N-dimensional component of an entangled state of an N- and an R-dimensional system ($R \leq N$) can be stored or transported by an M-dimensional system, so that the entangled state can be reconstructed (now, possibly with the two components spatially separated), with a mean fidelity of $F = (MR+1)/(NR+1)$. Teleportation of mixed states of rank R, can be done with the same fidelity.

The problem is closely linked to the issue of entanglement manipulation and quantum state transformation, and some of the above mentioned results can indeed be tested against special cases of the maximum fidelity of faithful transformation of a pure entangled state into a maximally entangled state of two N-level systems as computed by Vidal *et al* [1], Horodeckis [2]. The cited works identify the optimum theoretical transformation of the quantum channel, whereas our approach offers a different perspective as it deals with explicit operations on the incident quantum state.

We shall formulate our problem as the one of teleportation of an N-dimensional state through a perfect M-dimensional quantum channel. In Sec. II, we shall present our very simple scheme, and we shall present a calculation of its fidelity when applied to pure states. In Sec. III we compute the fidelity of teleportation of mixed states of the quantum system. In Sec. IV we summarize our conclusions and discuss implications of our results.

2 The Cutting Procedure

Assume that two parties, Alice and Bob, share a maximally entangled $M \times M$-dimensional state (the channel),

$$|\Psi_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i_A, i_B\rangle, \tag{1}$$

and that Alice possesses an arbitrary, unknown N-dimensional pure state ($N > M$) that she wants to transfer to a quantum system located at Bob's place with the greatest accuracy possible using only local quantum operations and classical communication. Transferring an N-dimensional quantum state $|\psi\rangle$ through an M-dimensional quantum channel cannot be done with unit fidelity if $M < N$ [3], but many different methods can be applied to do it approximately. What is the best teleportation scheme and what is the corresponding fidelity of the state which Bob recieves?

The method we are going to use is to first reduce the dimensionality of the state from N to M by a positive operator-valued measurement (POVM) [4], and by subsequently teleporting the resulting state perfectly through our M-dimensional quantum channel. If we choose the set $\{|\phi_j\rangle\}_{j=1}^{N}$ to form an orthonormal basis for the N-dimensional Hilbert-space of the initial state $|\psi\rangle$, the set of operators

$$\{\hat{A}_{\mathbb{i}}\}, \quad \hat{A}_{\mathbb{i}} = \frac{1}{\mathcal{N}} \sum_{j=1}^{M} |\phi_{i_j}\rangle\langle\phi_{i_j}| \tag{2}$$

constitutes our POVM that will be used to perform the $N \to M$ cut. The constant $\mathcal{N} = \binom{N-1}{M-1}$ can be determined from the normalisation condition, $\sum_{\mathbb{i}} \hat{A}_{\mathbb{i}} = \mathbb{1}$, and the sets of numbers $\mathbb{i} = \{i_1, \cdots, i_M\}$ runs through all the $\binom{N}{M}$ possible choices.

2.1 Pure States

We first consider the case of a pure initial state. The measurement outcome corresponding to $\hat{A}_{\mathbb{i}}$ occurs with probability $p_{\mathbb{i}} = \langle\psi|\hat{A}_{\mathbb{i}}|\psi\rangle$ in which case the projected state is $|\tilde{\psi}_{\mathbb{i}}\rangle = \frac{\hat{A}_{\mathbb{i}}|\psi\rangle}{\sqrt{\langle\psi|\hat{A}_{\mathbb{i}}{}^{\dagger}\hat{A}_{\mathbb{i}}|\psi\rangle}}$.

The fidelity of $|\tilde{\psi}_{\mathbb{i}}\rangle$ with respect to $|\psi\rangle$ is just the overlap

$$f_{\mathbb{i}} = |\langle\psi|\tilde{\psi}_{\mathbb{i}}\rangle|^2, \tag{3}$$

which we see is also equal to $f_{\mathbb{i}} = \mathcal{N}\langle\psi|\hat{A}_{\mathbb{i}}|\psi\rangle$. The average fidelity (averaged over measurement outcomes and over incident states) is therefore

$$\begin{aligned} F_{N\to M} &= \frac{1}{\mathcal{A}_N}\int d\Omega_N \sum_{\mathbb{i}} \mathcal{N}\langle\psi|\hat{A}_{\mathbb{i}}|\psi\rangle^2 \\ &= \frac{1}{\mathcal{A}_N}\int d\Omega_N \sum_{\mathbb{i}} \frac{1}{\mathcal{N}}\left(\sum_{j=1}^{M}|\langle\phi_{i_j}|\psi\rangle|^2\right)^2 \\ &= \frac{1}{\mathcal{A}_N}\int d\Omega_N \frac{1}{\mathcal{N}}\left[\mathcal{N}\frac{M-1}{N-1}\left(\sum_{j=1}^{N}|\langle\phi_j|\psi\rangle|^2\right)^2 + \left(\mathcal{N}-\mathcal{N}\frac{M-1}{N-1}\right)\sum_{j=1}^{N}|\langle\phi_j|\psi\rangle|^4\right] \\ &= \frac{M-1}{N-1} + \frac{N-M}{N-1}\frac{1}{\mathcal{A}_N}\int d\Omega_N \sum_{j=1}^{N}|\langle\phi_j|\psi\rangle|^4, \end{aligned} \tag{4}$$

where $d\Omega_N$ is the appropriate "surface area"-element on the unit hypersphere in the N-dimensional complex Hilbert space, and $\mathcal{A}_N \equiv \int d\Omega_N$. Since we average over input states, we do not need to average over different choices of the orthogonal basis. Equation (4) is therefore independent of the choice of basis states $\{|\phi_j\rangle\}$.

The integral in the last line in equation (4) we recognize as the average fidelity of estimating a state after a von Neumann measurement [4] in the basis $\{|\phi_j\rangle\}$

$$F_{N\to 1} = \frac{1}{\mathcal{A}_N}\int d\Omega_N \sum_{j=1}^{N}|\langle\phi_j|\psi\rangle|^4, \tag{5}$$

and we thus obtain the nice relation

$$F_{N\to M} = \frac{M-1}{N-1} + \frac{N-M}{N-1}F_{N\to 1}. \tag{6}$$

The problem is now reduced to that of calculating $F_{N\to 1}$, which is done in the following way. First we simplify equation (5) to

$$F_{N\to 1} = \frac{1}{\mathcal{A}_N} N \int d\Omega_N |\langle\phi_1|\psi\rangle|^4, \tag{7}$$

by noting that all N components of the state $|\psi\rangle = \sum_{j=1}^{N}\langle\phi_j|\psi\rangle|\phi_j\rangle$ will contribute equally to the sum after the averaging over states.

As a general representation for a state on the unit hypersphere in $\mathbb{C}^N$ we choose

$$|\psi\rangle = \begin{pmatrix} \cos\theta_1 e^{i\phi_1} \\ \sin\theta_1 \cos\theta_2 e^{i\phi_2} \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 e^{i\phi_3} \\ \vdots \quad\quad \vdots \quad\quad\quad \ddots \quad \ddots \\ \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \cos\theta_{N-1} e^{i\phi_{N-1}} \\ \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \sin\theta_{N-1} e^{i\phi_N} \end{pmatrix}, \qquad \begin{array}{c} 0 \le \theta_1, \dots, \theta_{N-1} \le \frac{\pi}{2} \\ 0 \le \phi_1, \dots, \phi_N \le 2\pi \end{array}, \tag{8}$$

and the corresponding measure, $d\Omega_N$, is found in the appendix to be

$$d\Omega_N = \prod_{k=1}^{N-1} \left(\cos\theta_k \sin\theta_k \left(\sin^2\theta_k\right)^{N-k-1} d\theta_k d\phi_k \right) d\phi_N, \tag{9}$$

The considered integral (7) can now be evaluated as

$$\begin{aligned} F_{N\to 1} &= \frac{N}{\mathcal{A}_N} \int d\Omega_N |\langle\phi_1|\psi\rangle|^4 = N \frac{\int d\Omega_N \cos^4\theta_1}{\int d\Omega_N} \\ &= N \frac{\int_0^{\frac{\pi}{2}} \cos\theta_1 \sin\theta_1 \left(\sin^2\theta_1\right)^{N-2} \cos^4\theta_1 d\theta_1}{\int_0^{\frac{\pi}{2}} \cos\theta_1 \sin\theta_1 \left(\sin^2\theta_1\right)^{N-2} d\theta_1} \\ &= \frac{2}{N+1}, \end{aligned} \tag{10}$$

and inserting this into the formula (6) yields the result

$$F_{N\to M} = \frac{M+1}{N+1}. \tag{11}$$

This value for the fidelity is in agreement with the following formula from the literature [2]

$$F_{N\to M}^{(\mathrm{opt})} = \frac{N f_s(\Psi_M) + 1}{N+1} \tag{12}$$

where $f_s(\Psi_M) = M/N$ is the *singlet fraction* of the channel, i.e. the fidelity by which the M-dimensional channel (1) can be transformed into an N-dimensional one (with which perfect teleportation can be subsequently achieved for any state in the N-dimensional Hilbert space). This agreement is reassuring, since for the task of teleportation we deal with the same shared quantum resources. By cutting the system to fit the resources rather than by extending the resources to fit the system, the present approach presents an alternative analysis to Ref. [2], and it treats simultaneously the tasks of teleportation and of storage or physical transport of a quantum state. The explicit calculation of the fidelity based on wave function overlaps also lends itself to further analysis, as we shall turn to in the section below and in the discussion.

3 Mixed States

The case of mixed states requires a special treatment. The appropriate measure of the fidelity of $\tilde{\rho}$ w.r.t. ρ is the *Bures fidelity* or *Uhlmann transition probability*, see e.g. [5, 6]

$$F(\rho,\tilde{\rho}) = \left(\mathrm{Tr}\,(\sqrt{\rho}\tilde{\rho}\sqrt{\rho})^{1/2}\right)^2, \tag{13}$$

which can also be written

$$F(\rho,\tilde{\rho}) = \max|\langle\phi|\tilde{\phi}\rangle|^2, \tag{14}$$

where the maximum is taken over all possible *purifications*, $|\phi\rangle$ and $|\tilde{\phi}\rangle$, of ρ and $\tilde{\rho}$ respectively. By a purification of a mixed state ρ acting on $\mathcal{H}$ is meant a pure state $|\xi\rangle \in \mathcal{H}\otimes\mathcal{H}_R$ fulfilling the condition $\rho = \mathrm{Tr}_R|\xi\rangle\langle\xi|$. If we write the Schmidt decomposition [4] of $|\xi\rangle$

$$|\xi\rangle = \sum_i \sqrt{\lambda_i}|i\rangle\otimes|i_R\rangle \tag{15}$$

we see that all the different purifications of ρ correspond to different choices of orthonormal basis sets $\{i_R\}$ for $\mathcal{H}_R$ (the λ_i's, i.e. the eigenvalues of ρ, are the same in all purifications). Since these are related by a unitary transformation, any purification can be found from a particular one by a transformation $|\xi\rangle \to (\mathbb{1}\otimes U)|\xi\rangle$, where U is unitary. Thus, if $|\phi_0\rangle$ and $|\tilde{\phi}_0\rangle$ are two particular purifications of ρ and $\tilde{\rho}$, the fidelity is

$$F(\rho,\tilde{\rho}) = \max|\langle\phi|\tilde{\phi}\rangle|^2 = \max_U|\langle\phi_0|(\mathbb{1}\otimes U)|\tilde{\phi}_0\rangle|^2, \tag{16}$$

where the maximum is now over unitary transformations U.

In the present situation we would like to teleport the mixed state

$$\rho = \sum_{ij}\rho_{ij}|\phi_i\rangle\langle\phi_j| \tag{17}$$

through the channel (1) with the above protocol. Thus we need to calculate the fidelity $f_{\mathbb{i}}$ of the teleported state $\tilde{\rho}_{\mathbb{i}} = \frac{\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^\dagger}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^\dagger)}$ w.r.t. the initial state ρ. We choose an arbitrary purification $|\psi\rangle$ of ρ and from this we construct a possible purification $|\tilde{\psi}_{\mathbb{i}}\rangle = \frac{(\hat{A}_{\mathbb{i}}\otimes\mathbb{1}_R)}{\sqrt{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^\dagger)}}|\psi\rangle$ of $\tilde{\rho}_{\mathbb{i}}$. Inserting these into equation (16) the particular fidelity is seen to be given by

$$f_{\mathbb{i}} = \max_U \frac{|\langle\psi|(\hat{A}_{\mathbb{i}}\otimes U)|\psi\rangle|^2}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^\dagger)}. \tag{18}$$

Letting

$$|\psi\rangle = \sum_{j=1}^N\sum_k c_{jk}|\phi_j,k\rangle, \tag{19}$$

where $|\phi_j, k\rangle$ is the tensor product of $|\phi_j\rangle$ in $\mathcal{H}$ and the $k^{\rm th}$ basis vector in $\mathcal{H}_R$, and using the expression (2) for the $\hat{A}_{\mathbb{i}}$'s this reduces to

$$
\begin{aligned}
f_{\mathbb{i}} &= \frac{1}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^{\dagger})} \max_U \left| \sum_{jj'} \sum_{kk'} c^*_{jk} c_{j'k'} \langle\phi_j|\hat{A}_{\mathbb{i}}|\phi_{j'}\rangle\langle k|U|k'\rangle \right|^2 \\
&= \frac{1}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^{\dagger})} \max_U \left| \frac{1}{\mathcal{N}} \sum_{j=1}^{M} \left(\sum_k c^*_{i_j k}\langle k| \right) U \left(\sum_k c_{i_j k}|k\rangle \right) \right|^2 \\
&= \frac{1}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^{\dagger})} \left| \frac{1}{\mathcal{N}} \sum_{j=1}^{M} \sum_k |c_{i_j k}|^2 \right|^2 \\
&= \frac{1}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^{\dagger})} \left| \frac{1}{\mathcal{N}} \sum_{j=1}^{M} \rho_{i_j i_j} \right|^2 \\
&= \frac{|\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho)|^2}{\mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho\hat{A}_{\mathbb{i}}^{\dagger})}.
\end{aligned}
\tag{20}
$$

The probability of the measurement outcome corresponding to $\hat{A}_{\mathbb{i}}$ is

$$
p_{\mathbb{i}} = \mathrm{Tr}(\hat{A}_{\mathbb{i}}\rho) = \langle\psi|(\hat{A}_{\mathbb{i}} \otimes \mathbb{1}_R)|\psi\rangle, \tag{21}
$$

and the expression for $f_{\mathbb{i}}$ simplifies to $f_{\mathbb{i}} = \frac{p_{\mathbb{i}}^2}{p_{\mathbb{i}}/\mathcal{N}} = \mathcal{N}p_{\mathbb{i}}$, and hence the teleportation fidelity is

$$
F_{N\to M} = \overline{\sum_{\mathbb{i}} p_{\mathbb{i}} f_{\mathbb{i}}}^{\psi} = \overline{\mathcal{N}\sum_{\mathbb{i}} \langle\psi|(\hat{A}_{\mathbb{i}} \otimes \mathbb{1}_R)|\psi\rangle^2}^{\psi}, \tag{22}
$$

where the overbar indicates averaging over all input states, ρ, performed by averaging over the purifications $|\psi\rangle$ of equation (19). Inserting the expression (2) for the $\hat{A}_{\mathbb{i}}$'s we find

$$
\begin{aligned}
F_{N\to M} &= \overline{\frac{1}{\mathcal{N}} \sum_{\mathbb{i}} \left(\sum_{j=1}^{M} \left(\sum_k |\langle\phi_{i_j}, k|\psi\rangle|^2 \right) \right)^2}^{\psi} \\
&= \overline{\frac{1}{\mathcal{N}} \left[\mathcal{N}\frac{M-1}{N-1} \left(\sum_{j=1}^{N} \sum_k |\langle\phi_j, k|\psi\rangle|^2 \right)^2 + \left(\mathcal{N} - \mathcal{N}\frac{M-1}{N-1} \right) \sum_{j=1}^{N} \left(\sum_k |\langle\phi_j, k|\psi\rangle|^2 \right)^2 \right]}^{\psi} \\
&= \frac{M-1}{N-1} + \frac{N-M}{N-1} F_{N\to 1}
\end{aligned}
\tag{23}
$$

exactly as in the pure state case, equation (6).

In evaluating $F_{N\to 1}$ we can again use the isotropy of the state space to get rid of a sum,

$$F_{N\to 1} = \overline{\sum_{j=1}^{N}\left(\sum_k |\langle\phi_j, k|\psi\rangle|^2\right)^2}^{\psi} = N\overline{\left(\sum_k |\langle\phi_1, k|\psi\rangle|^2\right)^2}^{\psi}. \tag{24}$$

The k-sum extends to $N_R = \dim \mathcal{H}_R$, but since we can always construct a purification by enlarging with a space of dimension $R = \mathrm{rank}(\rho)$, the states we need to average over can be chosen to be of the form $|\psi\rangle = |\psi^{(NR)}\rangle \in \mathcal{H}\otimes\mathbb{C}^R$. Hence

$$F_{N\to 1} = N\overline{\left(\sum_{k=1}^{R} |\langle\phi_1, k|\psi^{(NR)}\rangle|^2\right)^2}^{\psi} = N\frac{1}{\mathcal{A}_{NR}}\int d\Omega_{NR}\left(\sum_{k=1}^{R} |\langle\phi_1, k|\psi^{(NR)}\rangle|^2\right)^2. \tag{25}$$

Carrying out the square operation and applying the isotropy property and the representation (8) again, this expression reduces to

$$\begin{aligned} F_{N\to 1} &= \frac{N}{\mathcal{A}_{NR}}\int d\Omega_{NR}\left(\sum_{k=1}^{R} |\langle\phi_1, k|\psi^{(NR)}\rangle|^4 + \sum_{k<k'} 2|\langle\phi_1, k|\psi^{(NR)}\rangle|^2|\langle\phi_1, k'|\psi^{(NR)}\rangle|^2\right) \\ &= \frac{N}{\mathcal{A}_{NR}}\int d\Omega_{NR}\left(R|\langle\phi_1, 1|\psi^{(NR)}\rangle|^4 + \frac{R(R-1)}{2}2|\langle\phi_1, 1|\psi^{(NR)}\rangle|^2|\langle\phi_1, 2|\psi^{(NR)}\rangle|^2\right) \\ &= \frac{NR}{\mathcal{A}_{NR}}\int d\Omega_{NR}\left(\cos^4\theta_1 + (R-1)\cos^2\theta_1\sin^2\theta_1\cos^2\theta_2\right), \end{aligned} \tag{26}$$

which can be evaluated using the measure (9) as in the pure state case, and the result is

$$F_{N\to 1} = \frac{R+1}{NR+1} \tag{27}$$

for the mixed state state estimation fidelity. Inserting this into the formula (23) now also yields the mixed state teleportation fidelity,

$$F_{N\to M} = \frac{MR+1}{NR+1}. \tag{28}$$

4 Discussion

To summarize we have found a specific protocol with which the optimal fidelity is reached for teleportation of an N-dimensional state through an M-dimensional quantum channel or for storage in an M-dimensional system.

Based solely on the isotropic average over incident quantum states, we proved the relationship (6), (23)

$$F_{N\to M} = \frac{M-1}{N-1} + \frac{N-M}{N-1}F_{N\to 1} \tag{29}$$

between the fidelity $F_{N\to M}$ of the desired task and the state estimation fidelity $F_{N\to 1}$. The state estimation fidelity is computed by an explicit integration over the state space. Our most general result is obtained in the case where the quantum state of interest is the reduced density matrix for a pure state on the enlarged tensor product space $\mathcal{H}\otimes\mathbb{C}^R$. Such a reduced density matrix has rank less than or equal to the dimension R of the auxiliary space, and the fidelities $F_{N\to 1}$ and $F_{N\to M}$ depend on R (27), (28). Putting $R=1$ we thus obtain the result for pure states, known in the literature, see for example [7], and by chosing different values of R we obtain the fidelity for mixed states acting on $\mathcal{H}$, given the promise that their rank has the given value (the calculation in (26) involves averaging over states with rank less than or equal to R, but the states with smaller rank have zero measure in the integration).

The expression for $F_{N\to M}$ (28) shows that one may perform the operations in steps via states of intermediate dimensions $M<K<N$ without loss of fidelity $F_{N\to M}=F_{N\to K}F_{K\to M}$.

Our explicit calculation of the $F_{N\to 1}$ fidelities lend themselves to analyses where different promises are given about the incident state, leading to a change in the integration measure $d\Omega_N$. One may assign prior probability measures, for example restrict the calculations to real Hilbert spaces. As long as the isotropy is maintained our general formula (29) holds.

Let us comment on the dependence of fidelities on the mixed state character of the state. In the limit of very large N, a pure state is estimated with a probability scaling as $2/N$ (10), whereas a general mixed state with maximum rank, $R=N$, is estimated with half of that fidelity $1/N$ (27) which is the same as the fidelity of a pure guess of the state of the system. When observing the dependence of fidelities on the density matrix rank R, it should be remembered that these quantities are all computed under the assumption of a specific uniform state vector averaging over an enlarged space. For a given R, that space contains also density matrices with lower rank, but they have measure zero and hence they do not contribute to the average. It also contains states where the mixed state has only very small population on some of its components, in contrast to for example the maximally mixed state with $\rho=\frac{1}{N}\mathbb{1}$. If we were promised to have that particular density matrix, we could store and transmit that information classically with unit fidelity.

The example of transmitting a maximally mixed state is interesting, however, because it allows us to stress the important difference between the handling of quantum properties of the system alone, which could be done with unit fidelity, and for example the state of a system, which is entangled with some other quantum system. This latter case is encountered for example when teleportation serves in protocols for distributed quantum computing, [8, 9], see also [10]. Here it is clearly not sufficient to provide Bob with the classical description of the density matrix of Alice's subsystem. If the quantum state of systems A and Q is

$$|\psi_{AQ}\rangle=\frac{1}{\sqrt{N}}\sum_j|\phi_j,j_Q\rangle, \tag{30}$$

our protocol, which projects the A-system onto an M-dimensional subspace, and recreates that state in Bobs quantum system B, $|\tilde{\psi}_{\mathbb{1}}\rangle=\frac{1}{\sqrt{M}}\sum_j|\phi_{i_j}.i_{j_Q}\rangle$, has an average fidelity with the initial state of $F=M/N$.

Is the fidelity $F_{N\to M}$ (28) determined for mixed states in the previous section the fidelity for transmission of the N-dimensional Hilbert space component of an entangled state through an M-dimensional channel? To answer this question we note that according to (16), the Bures distance is obtained as the maximum wave function overlap with respect to unitary operations applied on the auxiliary Hilbert space, $F_{\text{mixed}} = \max_U |\langle\phi_0|(\mathbb{1}\otimes U)|\tilde{\phi}_0\rangle|^2$. The pure state fidelity, however, is obtained as the wave function overlap with no unitary operations applied. The calculation in Sec. 3 made no use of adjustments of U. Our results were obtained with $U = \mathbb{1}$, and since our fidelity measure *is* based on the wave function overlaps between pure states, Eq. (28) does indeed present the fidelity of the appropriate transformation of an entangled state. In particular, the maximally entangled state of two N-dimensional systems can have the state of one subsystem teleported to another location via an M-dimensional channel, so that the final entangled state is the correct one with a fidelity of $F = (MN+1)/(N^2+1)$.

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A Integration measure on a complex Hilbert space

We must determine the Jacobian required for changing between the two sets of complex cartesian and hyperspherical coordinates related by the transformation

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{N-1} \\ z_N \end{pmatrix} = \begin{pmatrix} r\cos\theta_1 e^{i\phi_1} \\ r\sin\theta_1\cos\theta_2 e^{i\phi_2} \\ r\sin\theta_1\sin\theta_2\cos\theta_3 e^{i\phi_3} \\ \vdots \quad \vdots \quad \ddots \quad \ddots \\ r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\theta_{N-1}e^{i\phi_{N-1}} \\ r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\theta_{N-1}e^{i\phi_N} \end{pmatrix}, \qquad \begin{matrix} 0\le r<\infty \\ 0\le\theta_1,\dots,\theta_{N-1}\le\frac{\pi}{2} \\ 0\le\phi_1,\dots,\phi_N\le 2\pi \end{matrix}. \tag{31}$$

For real polar coordinates

$$d(\rho\cos\alpha)d(\rho\sin\alpha) = \rho d\rho d\alpha, \tag{32}$$

and hence for a complex $z_1 = x_1 + ix_2 = r\cos\theta_1 e^{i\phi_1}$ we have

$$dx_1dx_2 = (r\cos\theta_1)d(r\cos\theta_1)d\phi_1. \tag{33}$$

For $N=2$, with $z_1 = x_1+ix_2$, $z_2 = x_3+ix_4$ we have

$$\begin{aligned} dx_1dx_2dx_3dx_4 &= (dx_1dx_2)(dx_3dx_4) = (r\cos\theta_1)d(r\cos\theta_1)d\phi_1(r\sin\theta_1)d(r\sin\theta_1)d\phi_2 \\ &= r^2\cos\theta_1\sin\theta_1 d(r\cos\theta_1)d(r\sin\theta_1)d\phi_1d\phi_2 = r^3\cos\theta_1\sin\theta_1 dr d\theta_1 d\phi_1 d\phi_2, \end{aligned} \tag{34}$$

and hence the Jacobian for $N=2$ is

$$\mathcal{J}_2(r,\theta_1) = r^3\cos\theta_1\sin\theta_1. \tag{35}$$

For a general N

$$\begin{aligned} &dx_1dx_2dx_3\cdots dx_{2N} = (dx_1dx_2)(dx_3\cdots dx_{2N}) \\ &= (r\cos\theta_1)d(r\cos\theta_1)d\phi_1\mathcal{J}_{N-1}(r\sin\theta_1,\theta_2,\theta_3,\dots,\theta_{N-1})d(r\sin\theta_1)d\theta_2d\theta_3\cdots d\theta_{N-1}d\phi_2\cdots d\phi_N \\ &= r^2\cos\theta_1\mathcal{J}_{N-1}(r\sin\theta_1,\theta_2,\theta_3,\dots,\theta_{N-1})drd\theta_1d\theta_2\cdots d\theta_{N-1}d\phi_1d\phi_2\cdots d\phi_N, \end{aligned} \tag{36}$$

and therefore

$$
\begin{aligned}
&\mathcal{J}_N(r,\theta_1,\theta_2,\ldots,\theta_{N-1}) = r^2\cos\theta_1\mathcal{J}_{N-1}(r\sin\theta_1,\theta_2,\theta_3,\ldots,\theta_{N-1})\\
&= r^2\cos\theta_1(\sin\theta_1)^{2(N-1)-1}\mathcal{J}_{N-1}(r,\theta_2,\theta_3,\ldots,\theta_{N-1})\\
&= \left(r^2\cos\theta_1\sin\theta_1(\sin^2\theta_1)^{N-2}\right)\left(r^2\cos\theta_2\sin\theta_2(\sin^2\theta_2)^{N-3}\right)\mathcal{J}_{N-2}(r,\theta_3,\theta_4\ldots,\theta_{N-1}) \qquad (37)\\
&= \prod_{k=1}^{N-2} r^2\cos\theta_k\sin\theta_k(\sin^2\theta_k)^{N-k-1}\mathcal{J}_2(r,\theta_{N-1}) = r^{2N-1}\prod_{k=1}^{N-1}\cos\theta_k\sin\theta_k(\sin^2\theta_k)^{N-k-1},
\end{aligned}
$$

where we have used equation (35) for $\mathcal{J}_2$, and the fact that $\mathcal{J}_k(\rho,\cdot)\propto\rho^{2k-1}$.

Equation (9) now follows by noting that

$$d\Omega_N = \left.\frac{dV_N}{dr}\right|_{r=1} = \mathcal{J}_N(1,\theta_1,\theta_2,\ldots,\theta_{N-1})d\theta_1 d\theta_2\cdots d\theta_{N-1}d\phi_1 d\phi_2\cdots d\phi_N. \qquad (38)$$

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